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Generalized quasilinearization for fractional differential equations

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ABSTRACT

In this paper, an existence and uniqueness result is obtained for an IVP of fractional differential equations using the method of generalized quasilinearization, which allows for some relaxation on the conditions on f.

1. Introduction

Quasilinearization [1] is a very useful iterative technique that guarantees the quadratic convergence of the sequence of the solutions of the corresponding linear problem. This has been extended to fractional differential equations in [2].

The technique can be refined to the situation where the function on the right hand side is split into two functions, one satisfying a weaker assumption than convexity and the other function satisfying a weaker assumption than concavity.

In this paper we extend this method to fractional differential equations which appear to be appropriate models for many physical phenomena and have attracted the attention of a number of research scientists [3–7].

2. Preliminaries

We start with the definitions of the Caputo fractional differential equation, the Riemann–Liouville fractional differential equation and the relations between the two derivatives.

The Caputo fractional differential equation is given by

$$^{c}D^{q}x = f(t, x), \quad x(t_0) = x_0 \tag{2.1}$$

and the corresponding Volterra fractional integral equation by

$$x(t) = x_0 + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s)) \mathrm{d}s. \tag{2.2}$$

The Riemann–Liouville fractional differential equation is expressed as

$$D^{q}x = f(t, x), \tag{2.3}$$

$$x(t_0) = x^0 = x(t)(t-t_0)^{1-q}|_{t=t_0} \tag{2.4}$$

and the corresponding Volterra fractional integral equation is

$$x(t) = x^0(t) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, x(s)) \mathrm{d}s, \tag{2.5}$$

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where

$$x^0(t) = \frac{x^0(t-t_0)^{q-1}}{\Gamma(q)}.$$

The relation between the Caputo fractional derivative and the Riemann–Liouville fractional derivative is as follows :

$$^cD^q x(t) = D^q\,[x(t) - x(t_0)]. \tag{2.6}$$

We shall state the needed notation and the required results from [8–10], without proof for the Riemann–Liouville fractional derivatives below.

Let $C_p\,([t_0,T],R) = \{u \in C((t_0,T],R)$ and $(t-t_0)^p\,u(t) \in C([t_0,T],R)\}$.

Lemma 2.1. *Let $m \in C_p\,([t_0,T],R)$ be locally Hölder continuous with exponent $\lambda > q$ and for any $t_1 \in (t_0,T]$, we have*

$$m(t_1) = 0 \quad \text{and} \quad m(t) \le 0 \text{ for } t_0 \le t \le t_1, \tag{2.7}$$

then

$$D^q m(t_1) \ge 0. \tag{2.8}$$

Lemma 2.2. *Let $\{x_\epsilon(t)\}$ be a family of continuous functions on $[t_0,T]$, for each $\epsilon > 0$, where*

$$D^q\,x_\epsilon(t) = f(t, x_\epsilon(t)), \qquad x_\epsilon^0 = x_\epsilon(t)(t-t_0)^{1-q}|_{t=t_0},$$

and

$$|f(t, x_\epsilon(t))| \le M \quad \text{for } t_0 \le t \le T.$$

Then the family $\{x_\epsilon(t)\}$ is equicontinuous on $[t_0,T]$.

The explicit solution of the nonhomogeneous linear fractional differential equation of the Caputo type is used to prove the main result. Hence we present it here. The nonhomogeneous linear fractional differential equation of the Caputo type is given by

$$^cD^q(x) = \lambda x + f(t), \qquad x(t_0) = x_0, \tag{2.9}$$

where $f \in C_q\,([t_0,T],R)$, and is Hölder continuous with exponent q.

Using the method of successive approximations we get the unique solution of (2.9) as

$$x(t) = x_0\,E_q(\lambda(t-t_0)^q) + \int_{t_0}^{t} (t-s)^{q-1}\,E_{q,q}(\lambda(t-s)^q)f(s)\mathrm{d}s, \quad t \in [t_0,T], \tag{2.10}$$

where

$$E_q(t) = \sum_{k=1}^{\infty} \frac{t^k}{\Gamma(qk+1)} \quad \text{and} \quad E_{q,q}(t) = \sum_{k=1}^{\infty} \frac{t^k}{\Gamma(qk+q)}$$

are the Mittag-Leffler functions of one parameter and two parameters respectively.

The next result deals with nonstrict fractional differential inequalities for the Caputo derivative.

Theorem 2.3. *Let $v, w \in C_p\,([t_0,T],R)$ be Hölder continuous for an exponent $0 < \lambda < 1$ and $\lambda > q$, $f \in C\,([t_0,T] \times R, R)$ and*

$$\text{(i)}\ ^cD^q\,v(t) \le f(t, v(t)), \qquad \text{(ii)}\ ^cD^q\,w(t) \ge f(t, w(t)), \quad t_0 < t \le T. \tag{2.11}$$

Suppose further that the standard Lipschitz condition

$$f(t,x) - f(t,y) \le L(x-y), x \ge y \quad \text{and} \quad L > 0, \tag{2.12}$$

is satisfied. Then $v(t_0) \le w(t_0)$ implies

$$v(t) \le w(t), \quad t_0 \le t \le T. \tag{2.13}$$

Corollary 2.4. *The function $f(t,v) = \sigma(t)v$ where $\sigma(t) \le L$ is admissible in Theorem 2.3 to yield $v(t) \le 0$ on $t_0 \le t \le T$.*

3. Generalized quasilinearization

To develop the method of generalized quasilinearization for the Caputo fractional differential equation we begin with the following details. The Caputo fractional differential equation is given by (2.1) and the corresponding Volterra fractional integral equation by (2.2). Here we consider the function $f(t,x)$ on the right hand side of (2.1) and split it into two parts as $f(t,x)$ and $g(t,x)$ where f satisfies a weaker condition than convexity and g satisfies a weaker assumption than concavity.

Consider the IVP, in this set up, given by

$$^cD^q x = f(t,x) + g(t,x), \quad x(t_0) = x_0, \tag{3.1}$$

The corresponding Volterra fractional integral equation is

$$x(t) = x_0 + \int_{t_0}^{t} (t-s)^{q-1}(f(s,x(s)) + g(s,x(s)))\mathrm{d}s. \tag{3.2}$$

We define the natural lower and upper solutions for the given equation (3.1).

Definition 3.1. $v, w \in C_p([t_0,T], R)$ are said to be lower and upper solutions, respectively, of (3.1) if the following hold.

$$^cD^q v \le f(t,v) + g(t,v), \quad v(t_0) \le x_0,$$
$$^cD^q w \ge f(t,w) + g(t,w), \quad w(t_0) \ge x_0, \; t \in J;$$

Theorem 3.2. *Assume that*

(i) $f, g \in C([t_0,T] \times R, R)$, $\alpha_0, \beta_0 \in C^q([t_0,T], R)$ *and*

$$^cD^q \alpha_0 \le f(t,\alpha_0) + g(t,\alpha_0), \quad \alpha_0(t_0) \le x_0$$
$$^cD^q \beta_0 \ge f(t,\beta_0) + g(t,\beta_0), \quad \beta_0(t_0) \ge x_0,$$
$$\alpha_0(t) \le \beta_0(t) \quad \text{on } J, \qquad \alpha_0(t_0) \le x_0 \le \beta_0(t_0), \quad \text{where } J = [t_0,T].$$

(ii) *Suppose $f_x(t,x)$ exists, $f_x(t,x)$ is increasing in x for each t,*

$$f(t,x) \ge f(t,y) + f_x(t,y)(x-y), \; x \ge y \quad \text{and}$$
$$|f_x(t,x) - f_x(t,y)| \le L_1|x-y|. \tag{3.3}$$

Further, suppose that $g_x(t,x)$ exists, $g_x(t,x)$ is decreasing in x for each t,

$$g(t,x) \ge g(t,y) + g_x(t,x)(x-y), \quad x \ge y$$

and

$$|g_x(t,x) - g_x(t,y)| \le L_2|x-y|.$$

Then there exist monotone sequences $\{\alpha_n\}, \{\beta_n\}$ such that $\alpha_n \to \rho$, $\beta_n \to r$ uniformly and monotonically to the unique solution $\rho = r = x$ of IVP (3.1) on J and the convergence is quadratic.

Proof. Consider the linear fractional differential equations given by

$$\begin{aligned} ^cD^q \alpha_{k+1} &= F(t,\alpha_{k+1},\alpha_k,\beta_k) \\ &= f(t,\alpha_k) + f_x(t,\alpha_k)(\alpha_{k+1}-\alpha_k) + g(t,\alpha_k) + g_x(t,\beta_k)(\alpha_{k+1}-\alpha_k), \\ \alpha_{k+1}(t_0) &= x_0 \end{aligned} \tag{3.4}$$

$$\begin{aligned} ^cD^q \beta_{k+1} &= G(t,\beta_{k+1},\alpha_k,\beta_k) \\ &= f(t,\beta_k) + f_x(t,\alpha_k)(\beta_{k+1}-\beta_k) + g(t,\beta_k) + g_x(t,\beta_k)(\beta_{k+1}-\beta_k) \\ \beta_{k+1}(t_0) &= x_0. \end{aligned} \tag{3.5}$$

Since the right hand sides of (3.4) and (3.5) satisfy a Lipschitz condition, it is clear that there exist unique solutions $\alpha_{k+1}(t)$ and $\beta_{k+1}(t)$, corresponding to (3.4) and (3.5) respectively. Our aim is to prove that

$$\alpha_0 \le \alpha_1 \le \cdots \le \alpha_k \le \beta_k \le \cdots \le \beta_1 \le \beta_0 \quad \text{on } J. \tag{3.6}$$

First, we will show

$$\alpha_0 \le \alpha_1 \le \beta_1 \le \beta_0 \quad \text{on } J, \tag{3.7}$$

by setting,

$$p = \alpha_0 - \alpha_1 \quad \text{on } J.$$

Then

$$^cD^qp = {}^cD^q\alpha_0 - {}^cD^q\alpha_1.$$

Thus, on using assumption (ii), we get

$$\begin{aligned} ^cD^qp &= {}^cD^q\alpha_0 - {}^cD^q\alpha_1 \\ &\le f(t,\alpha_0) + g(t,\alpha_0) - [f(t,\alpha_0) + f_x(t,\alpha_0)(\alpha_0-\alpha_1) + g(t,\alpha_0) + g_x(t,\beta_0)(\alpha_0-\alpha_1)] \\ &= [f_x(t,\alpha_0) + g_x(t,\beta_0)]p \end{aligned}$$

and $p(t_0) \le 0$.

Now applying Corollary 2.4, we obtain that

$$\alpha_0(t) \le \alpha_1(t) \quad \text{on } J.$$

Similarly, we obtain $\beta_1 \le \beta_0$ on J. Next write

$$p = \alpha_1 - \beta_1.$$

Then

$$\begin{aligned} ^cD^qp &= {}^cD^q\alpha_1 - {}^cD^q\beta_1 \\ &= f(t,\alpha_0) + g(t,\alpha_0) + f_x(t,\alpha_0)(\alpha_1-\alpha_0) + g_x(t,\beta_0)(\alpha_1-\alpha_0) \\ &\quad - [f(t,\beta_0) + g(t,\beta_0) + f_x(t,\alpha_0)(\beta_1-\beta_0) + g_x(t,\beta_0)(\beta_1-\beta_0)] \\ &\le f_x(t,\alpha_0)(\alpha_0-\beta_0) + f_x(t,\alpha_0)(\alpha_1-\alpha_0-\beta_1+\beta_0) + g_x(t,\beta_0)(\alpha_0-\beta_0) + g_x(t,\beta_0)(\alpha_1-\alpha_0-\beta_1+\beta_0) \\ &\le [f_x(t,\alpha_0) + g_x(t,\beta_0)](\alpha_1-\beta_1). \end{aligned}$$

This inequality holds by assumption (ii). Thus, we have

$$\begin{aligned} &^cD^qp \le (f_x(t,\alpha_0) + g_x(t,\beta_0))p \\ &p(t_0) = 0. \end{aligned}$$

Using Corollary 2.4, we conclude that $\alpha_1 \le \beta_1$ on J. Hence (3.7) is proved.

Assuming for some $k > 1$,

$$\alpha_0 \le \alpha_{k-1} \le \alpha_k \le \beta_k \le \beta_{k-1} \le \beta_0 \quad \text{on } J, \tag{3.8}$$

we shall show

$$\alpha_k \le \alpha_{k+1} \le \beta_{k+1} \le \beta_k \quad \text{on } J.$$

In order to do so, we set $p = \alpha_k - \alpha_{k+1}$ and get

$$\begin{aligned} ^cD^qp &= {}^cD^q\alpha_k - {}^cD^q\alpha_{k+1} \\ &= f(t,\alpha_{k-1}) + g(t,\alpha_{k-1}) + [f_x(t,\alpha_{k-1}) + g_x(t,\beta_{k-1})](\alpha_k-\alpha_{k-1}) \\ &\quad - \{f(t,\alpha_k) + g(t,\alpha_k) + [f_x(t,\alpha_k) + g_x(t,\beta_k)](\alpha_{k+1}-\alpha_k)\} \\ &\le f_x(t,\alpha_{k-1})(\alpha_{k-1}-\alpha_k) + f_x(t,\alpha_{k-1})(\alpha_k-\alpha_{k-1}) + f_x(t,\alpha_k)(\alpha_k-\alpha_{k+1}) + g_x(t,\beta_{k-1})(\alpha_{k-1}-\alpha_k) \\ &\quad + g_x(t,\beta_{k-1})(\alpha_k-\alpha_{k-1}) + g_x(t,\beta_k)(\alpha_k-\alpha_{k+1}) \\ &\le (f_x(t,\alpha_k) + g_x(t,\beta_k))p. \end{aligned}$$

This inequality holds by assumption (ii). Thus, we have

$$\begin{aligned} &^cD^qp \le (f_x(t,\alpha_k) + g_x(t,\beta_k))p \\ &p(0) = 0, \end{aligned}$$

by assumption (ii). Again, from Corollary 2.4, we conclude that

$$\alpha_k \le \alpha_{k+1}.$$

Similarly, we can show that $\beta_{k+1} \le \beta_k$ on J.

Finally, we show that $\alpha_{k+1} \le \beta_{k+1}$. To do so, set

$$p = \alpha_{k+1} - \beta_{k+1}.$$

Then,

$$\begin{aligned}
{}^cD^q p &= {}^cD^q \alpha_{k+1} - {}^cD^q \beta_{k+1} \\
&= f(t,\alpha_k) + g(t,\alpha_k) + [f_x(t,\alpha_k) + g_x(t,\beta_k)]\,(\alpha_{k+1} - \alpha_k) \\
&\quad - f(t,\beta_k) - g(t,\beta_k) - [f_x(t,\alpha_k) + g_x(t,\beta_k)]\,(\beta_{k+1} - \beta_k) \\
&\le f_x(t,\alpha_k)[\alpha_k - \beta_k + \alpha_{k+1} - \alpha_k - \beta_{k+1} + \beta_k] + g_x(t,\beta_k)\,[\alpha_k - \beta_k + \alpha_{k+1} - \alpha_k - \beta_{k+1} + \beta_k] \\
&= [f_x(t,\alpha_k) + g_x(t,\beta_k)]p.
\end{aligned}$$

Thus, we get ${}^cD^q p \le [f_x(t,\alpha_k) + g_x(t,\beta_k)]p$ and $p(t_0) = 0$, which yields, by Corollary 2.4, $\alpha_{k+1}(t) \le \beta_{k+1}(t)$ on J. Hence by induction principle we have that (3.6) is valid for all k.

Clearly the sequences are uniformly bounded because of (3.6), which shows that $\{{}^cD^q\alpha_n\}$, $\{{}^cD^q\beta_n\}$ are also uniformly bounded. By Lemma 2.2 we get the sequences are equicontinuous on $[t_0, T]$ and therefore using the Ascoli–Arzela Theorem we conclude that there is a subsequence that converges uniformly on $[t_0, T]$. This together with (3.6) gives that $\{\alpha_n\}$, $\{\beta_n\}$ converge uniformly and monotonically to ρ, r respectively as $n \to \infty$. Using the corresponding Volterra fractional integrals of (3.4) and (3.5) one can easily show that ρ and r are solutions of the IVP (3.1). Since $f_x(t,x)$ is bounded on the sector $[[\alpha_0, \beta_0] = \{x : \alpha_0(t) \le x \le \beta_0(t)\}]$, we obtain that $f(t,x)$ is Lipschitz and since $g(t,x)$ is decreasing, we find that $\rho = r = x$ is the unique solution of the IVP (3.1).

To prove the quadratic convergence of $\{\alpha_n\}$, $\{\beta_n\}$ to the unique solution, set $p_{n+1} = x - \alpha_{n+1}$. Then,

$$\begin{aligned}
{}^cD^q p_{n+1} &= {}^cD^q x - {}^cD^q \alpha_{n+1} \\
&= f(t,x) + g(t,x) - f(t,\alpha_n) - g(t,\alpha_n) - \{(f_x(t,\alpha_n) + g_x(t,\beta_n))[\alpha_{n+1} - \alpha_n]\} \\
&\le f_x(t,\xi)p_n + g_x(t,\eta)p_n + [f_x(t,\alpha_n) + g_x(t,\beta_n)]\,(p_{n+1} - p_n)
\end{aligned}$$

where $\alpha_n \le \xi \le x$ and $x \le \eta \le \beta_n$.

Now using the increasing nature of f_x and the decreasing nature of g_x, we get

$$\begin{aligned}
{}^cD^q p_{n+1} &\le (f_x(t,x) - f_x(t,\alpha_n))p_n + (g_x(t,x) - g_x(t,\beta_n))p_n + [f_x(t,\alpha_n) + g_x(t,\beta_n)]\,p_{n+1} \\
&\le L|p_n|^2 + [f_x(t,\alpha_n) + g_x(t,\beta_n)]\,p_{n+1} \\
&\le L|p_n|_0^2 + Kp_{n+1}
\end{aligned}$$

where

$$|p_n|_0 = \max_{[t_0,T]} |p_n(t)|,\ |f_x(t,\alpha_n)| \le K_1,\ |g_x(t,\beta_n)|, \le K_2$$

$$K = \max(K_1, K_2)$$

$$\text{and } L = \max\{L_1, L_2\}.$$

Thus we have,

$${}^cD^q p_{n+1} \le L|p_n|_0^2 + Kp_{n+1},$$

which gives

$$\begin{aligned}
p_{n+1}(t) &\le L|p_n|_0^2 \int_{t_0}^{t} (t-s)^{q-1} E_{q,q}(K((t-s)))\mathrm{d}s \\
&\le N|p_n|_0^2,
\end{aligned}$$

where

$$N = \frac{L}{q}(T - t_0)^q E_{q,q} \quad (K(T - t_0)^q).$$

Thus we have the estimate

$$|p_{n+1}|_0 \le N|p_n|_0^2,$$

which gives the quadratic convergence.

A similar computation shows that

$$|r_{n+1}|_0 \le N|r_n|_0^2.$$

The proof is therefore complete. □

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